The University of Texas at Austin
Dept. of Electrical and Computer Engineering
Final Exam

Date: August 15, 2016 Course: EE 313 Evans

Name: **Set ,** **Solution**
Last, First

- The exam is scheduled to last three hours.
- Open books and open notes. You may refer to your homework assignments and homework solution sets.
- **Power off all cell phones**
- You may use any standalone calculator or other computing system, i.e. one that is not connected to a network.
- Please do not wear hats or headphones during the exam.
- All work should be performed on the exam itself. If more space is needed, then use the backs of the pages.
- **Fully justify your answers unless instructed otherwise.**

Problem	Point Value	Your score	Topic
1	8		Continuous-Time Fourier Transform
2	12		Continuous-Time Frequency Response
3	8		Discrete-Time Fourier Transform
4	12		Discrete-Time Frequency Response
5	12		Discrete-Time Filter Design
6	12		Circuit Analysis
7	12		Convolution
8	12		Averaging Filters
9	12		Stability
Total	100		

Problem 1. Continuous-Time Fourier Transform. *8 points.*

The continuous-time Fourier transform transforms a continuous-time function $x(t)$ into a function $X(\omega)$ of a real-variable ω as follows:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$

(a) Using only the continuous-time Fourier transform definition above, find the Fourier transform of the continuous-time impulse $\delta(t)$. *4 points.*

$$\int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt = e^{-j\omega(0)} = 1$$

due to the sifting (sampling) property of the Dirac delta $\delta(t)$.

(b) Using only the continuous-time Fourier transform definition above, find the Fourier transform of a causal exponential signal $\exp(-t/\tau)\, u(t)$ where τ is the time constant in units of seconds where $\tau > 0$ and $u(t)$ is the unit step function. *4 points.*

$$\int_{-\infty}^{\infty} e^{-\frac{t}{\tau}} u(t)\, e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-\frac{t}{\tau}} e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-\left(\frac{1}{\tau} + j\omega\right)t}\, dt$$

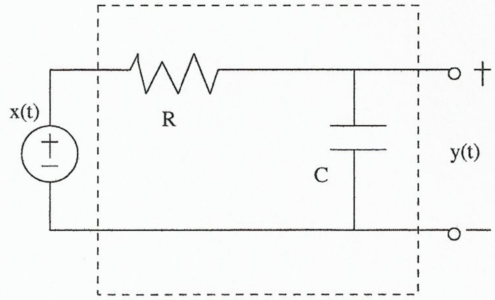
$$= -\frac{1}{\frac{1}{\tau} + j\omega}\, e^{-\left(\frac{1}{\tau} + j\omega\right)t} \Bigg|_{0}^{\infty}$$

$$= \underbrace{\lim_{t \to \infty} \frac{-1}{\frac{1}{\tau} + j\omega}\, e^{-\left(\frac{1}{\tau} + j\omega\right)t}} + \frac{1}{\frac{1}{\tau} + j\omega}$$

goes to zero as $t \to \infty$
because $\tau > 0$ and
$e^{-\frac{t}{\tau}} \to 0$ as $t \to \infty$.
$e^{j\omega t}$ oscillates as $t \to \infty$.

Problem 2. Continuous-Time Frequency Response. *12 points.*

Consider the following analog continuous-time circuit with input $x(t)$ and output $y(t)$:



From Midterm #2, Problem 2,

$$H(s) = \frac{\frac{1}{RC}}{s + \frac{1}{RC}} \quad \text{for } Re\{s\} > -\frac{1}{RC}$$

Also, $\tau = RC$

Analyze the circuit for $t > 0^-$ given that the initial voltage across the capacitor is 0 V.

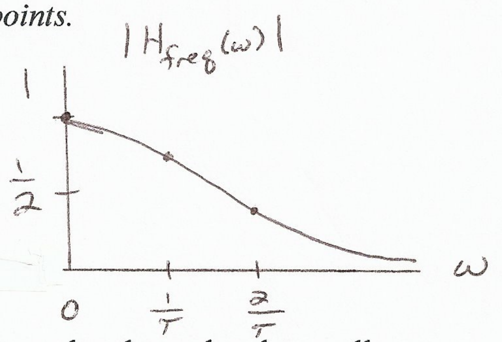
(a) Give a formula for the frequency response of the circuit. *3 points.*

Because the region of convergence for $H(s)$ includes the imaginary axis,

$$H_{freq}(\omega) = H(s)\Big|_{s=j\omega} = \frac{\frac{1}{RC}}{j\omega + \frac{1}{RC}} = \frac{\frac{1}{\tau}}{j\omega + \frac{1}{\tau}}$$

(b) Plot the magnitude of the frequency response of the circuit. *3 points.*

ω	$\lvert H_{freq}(\omega)\rvert$
0	1
$\frac{1}{\tau}$	$\frac{1}{\sqrt{2}} \approx 0.707$
$\frac{2}{\tau}$	$\frac{1}{\sqrt{5}} \approx 0.447$



(c) What is the frequency selectivity of the circuit? Lowpass, highpass, bandpass, bandstop, allpass or notch. Why? *3 points.*

Lowpass. Low frequencies pass through and high frequencies are attenuated.

(d) What is the bandwidth of the circuit? *3 points.*

#1 Cutoff frequency $= \frac{1}{\tau}$.

#2 Magnitude response falls below an arbitrary threshold of 0.2 at $\omega = \frac{5}{\tau}$.

Problem 3. Discrete-Time Signals. *8 points.*

The unilateral z-transform transforms a discrete-time function $x[n]$ into a function $X[z]$ of a complex-variable z as follows:

$$X[z] = \sum_{n=0}^{\infty} x[n] z^{-n}$$

The discrete-time Fourier transform transforms a discrete-time function $x[n]$ into a function $X_{freq}(\omega)$ of a real-variable ω as follows

$$X_{freq}(\omega) = \sum_{n=0}^{\infty} x[n] e^{-j\omega n}$$

Sometimes, the z-transform can be used to compute the discrete-time Fourier transform.

(a) Use the z-transform (definition, lookup tables, etc.) and appropriate additional work to find the discrete-time Fourier transform of the discrete-time impulse $\delta[n]$. *4 points.*

$$X[z] = \sum_{n=0}^{\infty} \delta[n] = 1 \text{ for all } z$$

Because the region of convergence includes the unit circle,

$$X_{freq}(\omega) = X[z]\Big|_{z=e^{j\omega}} = 1$$

(b) Use the z-transform (definition, lookup tables, etc.) and appropriate additional work to find the discrete-time Fourier transform of a causal exponential signal $a^n u[n]$ where a is a complex-valued constant and $|a| < 1$. 4 *points.*

$$X[z] = \mathcal{Z}\{a^n u[n]\} = \frac{1}{1 - a z^{-1}} \text{ for } |z| > |a|$$

Since $|a| < 1$, the region of convergence includes the unit circle, and

$$X_{freq}(\omega) = X[z]\Big|_{z=e^{j\omega}} = \frac{1}{1 - a e^{-j\omega}}$$

Problem 4. Discrete-Time Systems. *12 points.*

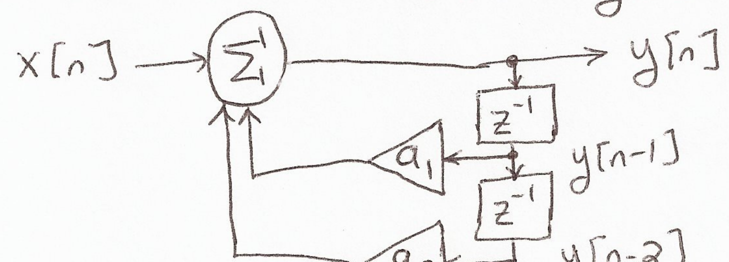
Consider a causal discrete-time linear time-invariant system with input $x[n]$ and output $y[n]$ being governed by the following difference equation

$$y[n] = (2\, r \cos \omega_0)\, y[n\text{-}1] - r^2\, y[n\text{-}2] + x[n]$$

where r is a real number $0 < r < 1$. Let $a_1 = 2r \cos \omega_0$ and $a_2 = -r^2$.

(a) Draw a block diagram for the difference equation. *3 points.*

Assume that we have $x[n]$ and $y[n]$:



(b) Please state all initial conditions. Please give values for the initial conditions to satisfy the stated system properties. *3 points.*

Let $n=0$: $y[0] = a_1 y[-1] + a_2 y[-2] + x[0]$

$n=1$: $y[1] = a_1 y[0] + a_2 y[-1] + x[1]$

$n=2$: $y[2] = a_1 y[1] + a_2 y[0] + x[2]$

Initial conditions and their values to satisfy LTI properties:

$y[-1] = 0$ and $y[-2] = 0$

(c) Find the equation for the transfer function $H[z]$ in the z-domain, including the region of convergence. *3 points.*

$$Y[z] = a_1 z^{-1} Y[z] + a_2 z^{-2} Y[z] + X[z]$$

$$(1 - a_1 z^{-1} - a_2 z^{-2})\, Y[z] = X[z]$$

$$H[z] = \frac{Y[z]}{X[z]} = \frac{1}{1 - a_1 z^{-1} - a_2 z^{-2}} \text{ for } |z| > r$$

Poles are at $re^{j\omega_0}$ and $re^{-j\omega_0}$ from Problem 5

(d) Find a formula for the frequency response for the system. *3 points.*

Since the region of convergence includes the unit circle, because $0 < r < 1$,

$$X_{freq}(\omega) = X[z]\Big|_{z=e^{j\omega}} = \frac{1}{1 - a_1 e^{-j\omega} - a_2 e^{-j2\omega}}$$

Problem 5. Discrete-Time Filter Design. *12 points.*

We are going to design parameters for the system in Problem 4.

The system is a causal discrete-time linear time-invariant system with input $x[n]$ and output $y[n]$ being governed by the following difference equation

$$y[n] = (2\, r \cos \omega_0)\, y[n-1] - r^2\, y[n-2] + x[n]$$

where r is a real number $0 < r < 1$. The poles are located at $z = r \exp(j\, \omega_0)$ and $z = r \exp(-j\, \omega_0)$.

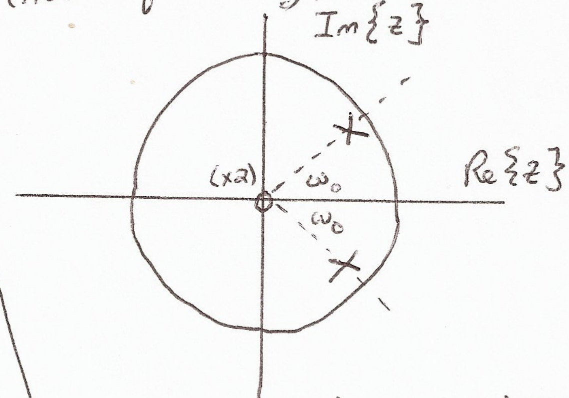
The signal $x[n]$ results from sampling a continuous-time signal $x(t)$ at a sampling rate of f_s.

The filter should pass the continuous-time frequency f_0 in Hz and attenuate as many of the other frequencies as possible.

(a) What are all of the posible values of f_0? *3 points.*

From the sampling theorem, $f_s > 2 f_0 \Rightarrow f_0 < \frac{1}{2} f_s$

Pole-zero diagram (not required by problem)



- Poles at $re^{j\omega_0}$ and $re^{-j\omega_0}$ where $0 < r < 1$
- Zeros at 0 and 0.

(b) Determine the value of ω_0. *3 points.*

$$\omega_0 = 2\pi \frac{f_0}{f_s}$$

(c) Determine the value of r where $0 < r < 1$. *3 points.*

Poles closer to the unit circle, but still inside the unit circle, give a stronger passband response. $r = 0.95$

(d) If $r = 1$, give a formula for the output $y[n]$ if $x[n] = \delta[n]$. *3 points.*

$$y[n] = (2 r \cos \omega_0)\, y[n-1] - r^2 y[n-2] + \delta[n]$$

From Problem 4,

$$H(z) = \frac{1}{1 - (2\cos\omega_0) z^{-1} + z^{-2}} \quad \text{for } |z| > 1$$

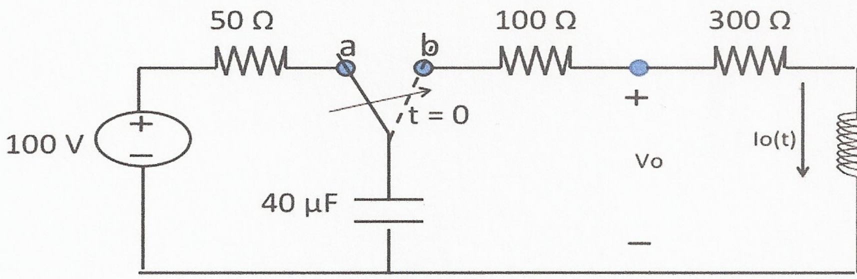
$$H(z) = \frac{z^{-1}}{\sin\omega_0} \cdot \frac{z \sin\omega_0}{z^2 - (2\cos\omega_0) z + 1}$$

$$h[n] = \frac{1}{\sin\omega_0} \sin(\omega_0 (n-1))\, u[n-1]$$ using entry 11b in Lathi Table 5.1 on page 498.

Problem 6. Circuit Analysis. *12 points.*

The switch in the circuit below has been in position 'a' for a long time.

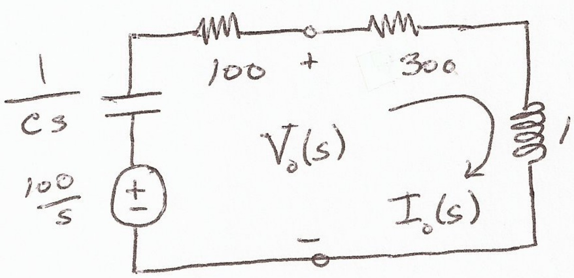
At t=0, the switch is moved to position 'b'.



Determine $v_o(t)$ by taking the Laplace transform of the circuit, and then solving for the time-domain expression.

Because the switch was in position 'a' for a long time, and the voltage source is a constant 100 V, the voltage across the capacitor will be 100 V when the switch is moved to position 'b'. The initial current across the capacitor will be zero.

The Laplace transform of the circuit is shown below:



$Z(s) = \frac{1}{Cs} + 400 + Ls$

$Z(s) = \frac{25000}{s} + 400 + 10s$

$V_o(s) = \frac{300 + 10s}{\frac{25000}{s} + 400 + 10s} V(s)$

$c = 40\mu F; \quad \frac{1}{c} = 25000$

$V(s) = \frac{100}{s}$; Voltage Divider; $V_o(s) = \frac{300 + 10s}{25000 + 400s + 10s^2} \cdot 100$

Using Lathi, Table 4.1, entry 10c, p. 344 $\Leftarrow$ $V_o(s) = \frac{3000 + 100s}{2500 + 40s + s^2}$

$\frac{As + B}{s^2 + 2as + c}$ where $A = 100$, $B = 3000$, $a = 20$, $c = 2500$,

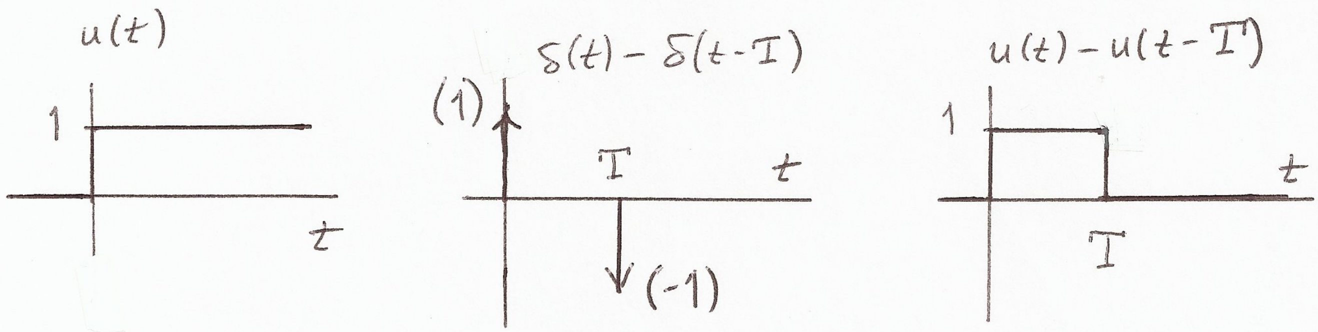
$V_o(t) = 102.35\, e^{-20t} \cos(45.8t - 12.31°)\, u(t)$

Problem 7. Convolution. *12 points.*

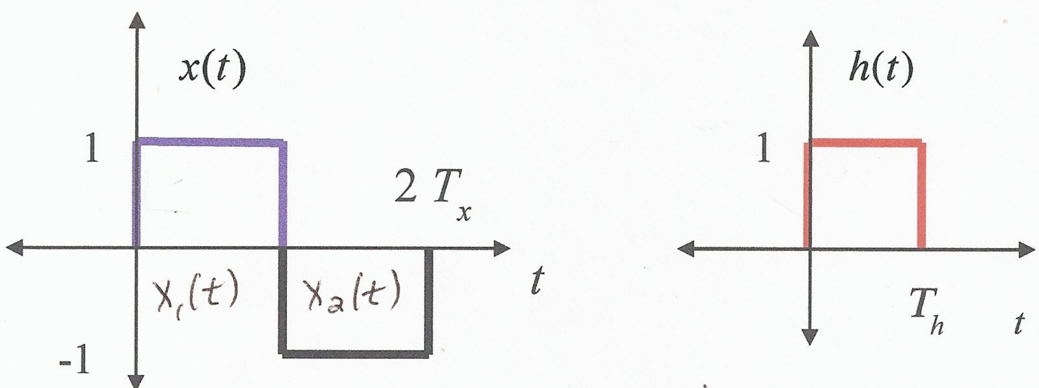
Please solve the following two convolution problems and explain the method you have chosen to use.

(a) Convolve the unit step function $u(t)$ with the signal $\delta(t) - \delta(t - T)$ where $T > 0$. Plot all three signals: $u(t)$ and $\delta(t) - \delta(t - T)$ and their convolution. *6 points.*

$$u(t) * (\delta(t) - \delta(t-T)) = u(t) * \delta(t) - u(t) * \delta(t-T)$$
$$= u(t) - u(t-T)$$



(b) Convolve the following continuous-time signals, assuming that $T_h < T_x$. Plot the result. *6 points.*



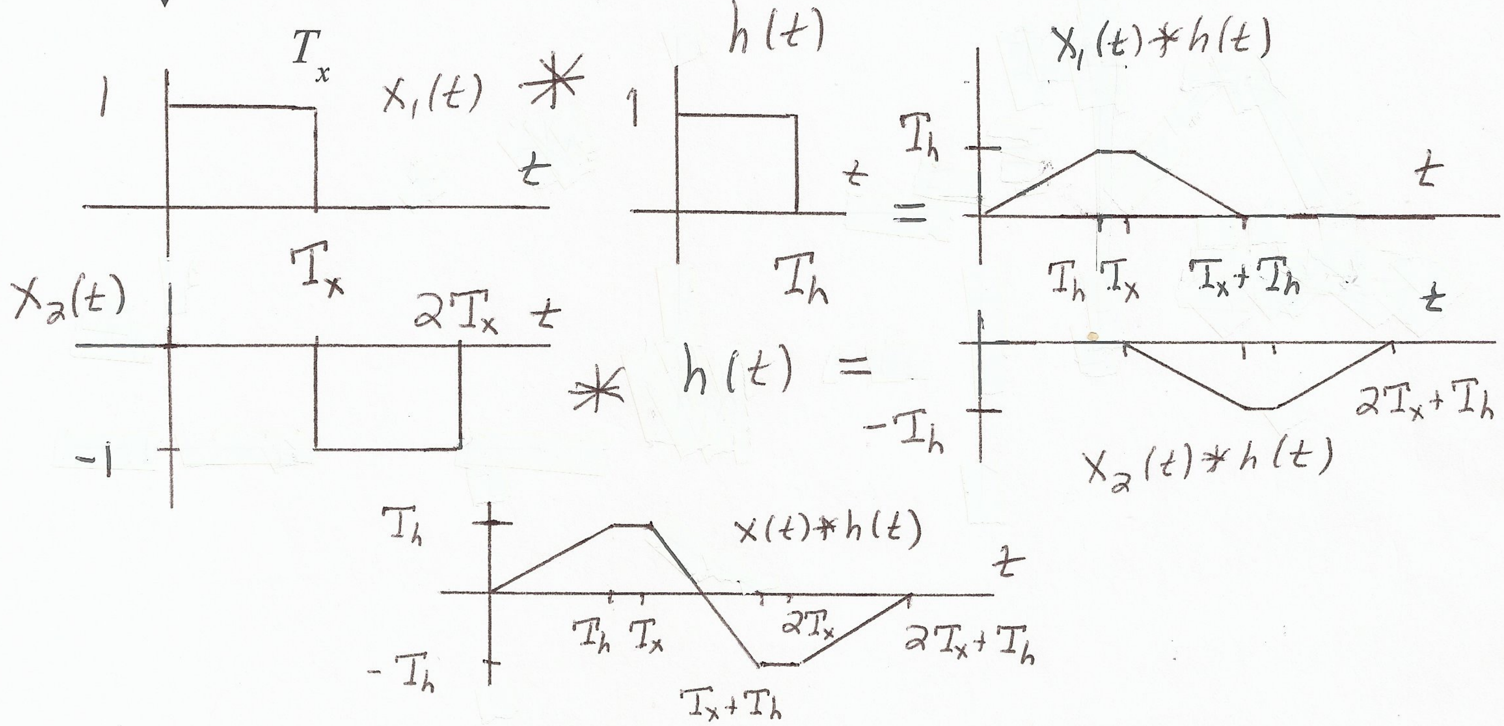
Convolution:

$x(t) = x_1(t) + x_2(t)$

$x(t) * h(t) =$

$x_1(t) * h(t) +$

$x_2(t) * h(t)$



Problem 8. Averaging Filters. *12 points.*

An averaging filter is used to reduce noise and smooth out data from one value to the next.

A two-coefficient averaging filter with input $x[n]$ and output $y[n]$ is

$$y[n] = \frac{1}{2}x[n] + \frac{1}{2}x[n-1] = \frac{x[n] + x[n-1]}{2}$$

That is, the output is the average value of the current input value and the previous input value.

The three-coefficient averaging filter with input $x[n]$ and output $y[n]$ is

$$y[n] = \frac{1}{3}x[n] + \frac{1}{3}x[n-1] + \frac{1}{3}x[n-2] = \frac{x[n] + x[n-1] + x[n-2]}{3}$$

Please answer the following questions about the **three-coefficient averaging filter**.

(a) What are the initial conditions of the averaging filter? What values should they have in order for the averaging filter to have the system properties of linearity and time-invariance? *3 points.*

Let $n=0$: $y[0] = \frac{1}{3}x[0] + \frac{1}{3}x[-1] + \frac{1}{3}x[-2]$

$n=1$: $y[1] = \frac{1}{3}x[1] + \frac{1}{3}x[0] + \frac{1}{3}x[-1]$

Initial conditions and their values are $x[-1] = 0$ and $x[-2] = 0$

(b) Give a formula for the impulse response of the averaging filter. Plot the impulse response. Is the impulse response of finite duration or infinite duration? *3 points.*

Let $x[n] = \delta[n]$.

Response $y[n] = h[n] = \frac{1}{3}\delta[n] + \frac{1}{3}\delta[n-1] + \frac{1}{3}\delta[n-2]$

(c) What is the transfer function in the z-domain? Please include the region of convergence. *3 points.*

$H(z) = \mathcal{Z}\{h[n]\} = \frac{1}{3} + \frac{1}{3}z^{-1} + \frac{1}{3}z^{-2}$ for $z \neq 0$

(d) Plot the magnitude response. What kind of frequency selectivity does this filter have? Lowpass, highpass, bandpass, bandstop, allpass or notch? *3 points.*

Because the region of convergence includes unit circle,

$H_{freq}(\omega) = H(z)\Big|_{z=e^{j\omega}} = \frac{1}{3} + \frac{1}{3}e^{-j\omega} + \frac{1}{3}e^{-j2\omega}$

Lowpass filter. Reduces noise in high frequencies.

ω	$\lvert H_{freq}(\omega)\rvert$
0	1
$\frac{\pi}{3}$	$\frac{2}{3}$
$\frac{2\pi}{3}$	0
π	$\frac{1}{3}$

Problem 9. Stability. *12 points.*

Consider a system with input $x(t)$ and output $y(t)$ governed by the differential equation

$$y''(t) + y(t) = x'(t) - x(t)$$

for $t > 0^-$.

(a) What are the characteristic roots of the system? *3 points.*

$\lambda^2 + 1 = 0 \Rightarrow \lambda^2 = -1 \Rightarrow \lambda = j$ and $\lambda = -j$

(b) What are the initial conditions of the system? *3 points.*

$y'(0^-)$, $y(0^-)$ and $x(0^-)$

(c) Assume that the initial conditions are not zero. Is the zero-input response asymptotically unstable, marginally stable or stable? *3 points.*

Characteristic roots	Asymptotic Stability
at least one right-hand plane	unstable
or repeated root imaginary axis	unstable
or at least single root imag. axis	marginally stable $\Leftarrow \lambda = \pm j$
or otherwise	stable

(d) Assume that the initial conditions are zero. What is the transfer function in the Laplace domain? Is the system bounded-input bounded-output (BIBO) stable? Why or why not? *3 points.*

$$(s^2 + 1) Y(s) = (s - 1) X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s - 1}{s^2 + 1}$$

poles are at $+j$ and $-j$ $\Rightarrow$ BIBO unstable

zero is at 1